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On Some Improperly Posed Problems for the Chaplygin Equation*

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1. INTRODUCTION

The solutions of a number of physically important problems may be characterized by solutions of partial differential equations of mixed type. One of the best known examples is that of the isentropic, irrotational transonic flow of a compressible fluid. In the hodograph plane the governing equation for the stream function is the so-called Chaplygin equation (see e.g. Bers [1], Bitŝadze [2])

$$\frac{\partial^2 u}{\partial y^2} + h(y) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.1}$$

Since it is extremely difficult to pose in the hodograph plane a mathematical problem which corresponds to a prescribed flow problem in the physical plane, the inverse method has been widely adopted; i.e., one formulates a reasonable problem for (1.1) in the hodograph plane and then checks to see whether in the physical flow plane the problem is meaningful. This has led to a rather careful study of a number of specific boundary value problems for (1.1), for instance the (i) Frankl problem, (ii) Tricomi problem, etc. (see, e.g., Bers [1], Schiffer [3], Bitŝadze [2]). Some of these problems lead to meaningful physical problems, but many of them do not. Nevertheless they are of considerable interest in themselves and perhaps will prove of importance in some other connection in the future.

It is in the spirit of this last remark that we consider other types of initial-boundary value problems for the Chaplygin equation (1.1). In Section 2 we treat an initial-boundary value problem for a rectangular domain $D : \{0 < x < 1, -y_0 < y < Y\}$ in which the equation may change type: *Problem A* Cauchy data is prescribed on $y = -y_0$ and Dirichlet data is

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prescribed on $x = 0$ and $x = 1$. It can easily be seen that unless some restriction is placed on the class of admissible solutions the problem is not well posed in the sense of Hadamard; i.e., the solution will not depend continuously on the data. On the other hand, any restrictive condition which is imposed in order to effect stability should hopefully be physically realizable. The restriction which we impose is that u should belong to the class $\mathcal{M}$ of solutions of (1.1) whose L_2 integrals over D are bounded by some prescribed constant M. Under the assumption that there exists a constant $\alpha \geqslant 0$ such that

$$h'(y) - \alpha h(y) \geqslant 0 \qquad -y_0 \leqslant y \leqslant Y, \tag{1.2}$$

it is shown that this restriction on u guarantees stability in L_2 and, in fact, leads to a method for obtaining error bounds.

In Section 3 we exhibit a specific example in which condition (1.2) is violated and show that the restriction that u belong to $\mathcal{M}$ is not sufficient to guarantee stability for Problem A.

In the particular case of transonic flow where $h(y) = y$ and $\alpha = 0$, in (1.1) and (1.2), the results of Sections 2 and 3 may be interpreted as follows. If Cauchy data is prescribed on $y = -y_0$ (i.e., on a line in the region corresponding to supersonic flow) and u belongs to $\mathcal{M}$, then u depends continuously in L_2 on the Cauchy data. On the other hand, if Cauchy data is prescribed on $y = Y$ (i.e., on a line in the region corresponding to subsonic flow), then the restriction that u belong to $\mathcal{M}$ is not sufficient to guarantee continuous dependence on the Cauchy data.

Finally, in Section 4 the stability in L_2 of an improperly posed initial-boundary value problem for the equation conjugate to the Chaplygin equation is discussed.

It is of course well known (see, e.g., Pucci [4] and John [5]) that in treating improperly posed problems one must allow for error in measurement of the data. Methods for handling such error have been discussed by Payne [6], Trytten [7], Schaefer [8], etc. For further references to stability study for improperly posed problems see a forthcoming paper by Payne [9]. It should be remarked that up to now there appear to have been no stability investigations for equations of mixed type.

In this paper we consider only a single linear equation of mixed type in two independent variables. Some nonlinear equations of mixed type in n independent variables will be treated in a subsequent paper.

2. In this section we consider the following initial-boundary value problem:

PROBLEM A.

$$Lu \equiv u_{,yy} + h(y)\, u_{,xx} = 0 \qquad \text{in} \qquad D, \tag{2.1}$$

$$u \text{ prescribed on } \quad x = 0 \quad \text{and} \quad x = 1, \tag{2.2}$$

$$u, \frac{\partial u}{\partial y} \quad \text{prescribed on} \quad y = -y_0 \quad (y_0 \geqslant 0). \tag{2.3}$$

Here D is the rectangular domain $0 < x < 1$ and $-y_0 < y < Y$ $(Y \geqslant 0)$. (In (2.1) and what follows the comma followed by a subscript denotes differentiation). We assume that the function h is continuously differentiable in $\bar{D}$ and that there exists a constant $\alpha \geqslant 0$ such that

$$h'(y) - \alpha h(y) \geqslant 0 \tag{2.4}$$

for all y satisfying $-y_0 \leqslant y \leqslant Y$. Let us note that condition (2.4) is satisfied, in particular, if $h'(y) \geqslant 0$ and $\alpha = 0$.

We shall say that u belongs to $\mathscr{M}$ if the condition

$$\iint_D u^2 \, dx \, dy \leqslant M^2 \tag{2.5}$$

is satisfied for some prescribed constant M. We proceed to derive *a priori* inequalities which for solutions u of Problem A that belong to $\mathscr{M}$ may be used to obtain uniqueness, continuous dependence on the data and L_2 bounds on compact subdomains of D. The basic inequalities are derived under the additional assumption that $\alpha > 0$ in (2.4) and in a form which yields immediately the desired L_2 bounds; by specialization, these inequalities imply also the properties of uniqueness and continuous dependence on the data when α is nonnegative.

Let us approximate u by a function φ which is piecewise C^2 in D, C^1 in $\bar{D}$ and such that $\varphi = u$ on $x = 0$ and $x = 1$. We set

$$v = u - \varphi \tag{2.6}$$

and let D_y denote the rectangular domain $0 < x < 1$ and $-y_0 < \eta < y \leqslant Y$. Let $\mathscr{F}$ denote the functional

$$\mathscr{F}(v, y) = \log F(v, y) + cy^2, \qquad -y_0 < y < Y, \tag{2.7}$$

where

$$\begin{aligned} F(v, y) = \iint_{D_y} v^2 \, dx \, d\eta + (Y - y) \int_0^1 \underset{y=-y_0}{v^2} \, dx \\ + k_1 \int_0^1 \underset{y=-y_0}{v^2} \, dx + k_2 \int_0^1 \underset{y=-y_0}{(v^2{}_{,x} + v^2{}_{,y})} \, dx \\ + k_3 \iint_D (L\varphi)^2 \, dx \, dy \equiv \iint_{D_y} v^2 \, dx \, d\eta + Q^2. \end{aligned} \tag{2.8}$$

Here the constants c and k_i in (2.7) and (2.8) are to be determined. We note that Q involves only data terms of $u - \varphi$ and, hence, for fixed k_i it will be small if the data of φ approximates the data of u sufficiently well.

Our *a priori* inequalities are a consequence of the following convexity argument. If we can establish that $\mathscr{F}$ is a convex function of y then an application of Jensen's inequality yields

$$e^{cy^2}\left[\iint_{D_y} v^2\,dx\,d\eta + Q^2\right] \leqslant \left\{e^{cY^2}\left[\iint_D v^2\,dx\,d\eta + Q^2\right]\right\}^{\frac{y+y_0}{y_0+Y}} \cdot \left\{e^{cy_0^2}Q^2\right\}^{\frac{Y-y}{y_0+Y}}. \tag{2.9}$$

Since φ is at our disposal and u is to be in class $\mathscr{M}$ we may assume without loss of generality that

$$\iint_D v^2\,dx\,dy \leqslant M_0^2. \tag{2.10}$$

Moreover, since Q involves only data terms we have

$$Q^2 \leqslant \nu M_0^2 \tag{2.11}$$

for some computable constant ν. Therefore

$$\iint_{D_y} v^2\,dx\,d\eta + Q^2 \leqslant e^{c(y_0+y)(Y-y)}\{(1+\nu)M_0^2\}^{\frac{y+y_0}{y_0+Y}} Q^{\frac{2(Y-y)}{y_0+Y}}, \tag{2.12}$$

from which follows the desired L_2 bounds for u on compact subdomains of D.

Let us now show that $\mathscr{F}$ is a convex function of y. We note that

$$F^2\mathscr{F}'' = FF'' - (F')^2 + 2cF^2, \tag{2.13}$$

where the prime denotes differentiation with respect to y. Hence, in order to establish the convexity of $\mathscr{F}$ it is sufficient to show that the right-hand side of (2.13) is positive. We note that

$$F' = 2\iint_{D_y} vv_{,\eta}\,dx\,d\eta, \tag{2.14}$$

$$F'' = 2\iint_{D_y} v^2_{,\eta}\,dx\,d\eta + 2\iint_{D_y} vv_{,\eta\eta}\,dx\,d\eta + 2\int_{\substack{0\\ y=-y_0}}^1 vv_{,y}\,dx. \tag{2.15}$$

Moreover,

$$\iint_{D_y} vv_{,\eta\eta}\,dx\,d\eta = \iint_{D_y} hv^2_{,x}\,dx\,d\eta - \iint_{D_y} vL\varphi\,dx\,d\eta \tag{2.16}$$

and, hence,

$$F'' = 4 \iint_{D_y} v^2{}_{,\eta}\, dx\, d\eta + 2 \iint_{D_y} (-v^2{}_{,\eta} + hv^2{}_{,x})\, dx\, d\eta$$

$$- 2\iint_{D_y} vL\varphi\, dx\, d\eta + 2 \int_{0}^{1}\Big|_{y=-y_0} vv_{,y}\, dx. \tag{2.17}$$

A lower bound for the second term in (2.17) is required in the sequel. We note that $(-y_0 < t < y)$

$$0 = 2 \iint_{D_t} (t - \eta)\, v_{,\eta}\{v_{,\eta\eta} + hv_{,xx} + L\varphi\}\, dx\, d\eta$$

$$= 2 \iint_{D_t} (t - \eta)\, v_{,\eta} L\varphi\, dx\, d\eta - (y + y_0) \int_{0}^{1}\Big|_{y=-y_0} (v^2{}_{,y} - hv^2{}_{,x})\, dx$$

$$+ \iint_{D_t} (t - \eta)\, h' v^2{}_{,x}\, dx\, d\eta$$

$$+ \iint_{D_t} (v^2{}_{,\eta} - hv^2{}_{,x})\, dx\, d\eta. \tag{2.18}$$

Applying the arithmetic-geometric mean inequality to the first term in (2.18) and using (2.4) with $\alpha > 0$, we obtain

$$\frac{dJ(t)}{dt} \geqslant I + \alpha J(t), \tag{2.19}$$

where

$$I = -(y_0 + Y)\Big[\int_{0}^{1}\Big|_{y=-y_0} (v^2{}_{,y} + |h|\, v^2{}_{,x})\, dx + \alpha^{-1} \iint_{D} (L\varphi)^2\, dx\, dy\Big], \tag{2.20}$$

$$J(t) = \iint_{D_t} (t - \eta)(-v^2{}_{,\eta} + hv^2{}_{,x})\, dx\, d\eta. \tag{2.21}$$

Integrating (2.19) from $-y_0$ to y, we find

$$\iint_{D_y} (-v^2{}_{,\eta} + hv^2{}_{,x})\, dx\, d\eta \geqslant -Ie^{\alpha(y+y_0)}. \tag{2.22}$$

If we now substitute (2.22) into (2.17) we obtain for some computable constants c_i

$$FF'' - (F')^2 \geqslant 4G + 4Q^2 \iint_{D_v} v^2{}_{,\eta}\, dx\, d\eta$$
$$- F \left\{ c_1 \iint_{D_v} v^2\, dx\, d\eta + c_2 \int_{\substack{0\\ y=-y_0}}^{1} v^2\, dx + c_3 \int_{\substack{0\\ y=-y_0}}^{1} v^2{}_{,y}\, dx \right.$$
$$\left. + c_4 \int_{\substack{0\\ y=-y_0}}^{1} v^2{}_{,x}\, dx + c_5 \iint_D (L\varphi)^2\, dx\, dy \right\}, \qquad (2.23)$$

where

$$G = \left(\iint_{D_v} v^2\, dx\, d\eta \right) \left(\iint_{D_v} v^2{}_{,\eta}\, dx\, d\eta \right) - \left(\iint_{D_v} v v_{,\eta}\, dx\, d\eta \right)^2. \qquad (2.24)$$

Since $G \geqslant 0$ by Schwarz's inequality it follows that

$$F^2 \mathscr{F}'' \geqslant 2cF^2 - F \left\{ c_1 \iint_{D_v} v^2\, dx\, d\eta + c_2 \int_{\substack{0\\ y=-y_0}}^{1} v^2\, dx \right.$$
$$\left. + c_3 \int_{\substack{0\\ y=-y_0}}^{1} v^2{}_{,y}\, dx + c_4 \int_{\substack{0\\ y=-y_0}}^{1} v^2{}_{,x}\, dx + c_5 \iint_D (L\varphi)^2\, dx\, dy \right\}. \qquad (2.25)$$

Obviously the constants k_i in Q and c can be chosen so that the right hand side of (2.25) is positive. In fact for any positive k_i we need only choose c sufficiently large so that the first term in (2.25) is the dominant one. Hence $\mathscr{F}$ is a convex function of y and we have established the following result:

THEOREM 1. *Let u be a solution of Problem A which belongs to class $\mathscr{M}$. Suppose that h satisfies (2.4) for some positive α. Let v, Q and M_0 be defined as in (2.6), (2.8), and (2.10), respectively. Then there exists a computable constant K such that*

$$\iint_{D_v} v^2\, dx\, d\eta \leqslant K M_0^{\frac{2(v+y_0)}{y_0+Y}} \underset{\sim}{Q}^{\frac{2(Y-v)}{y_0+Y}}, \qquad -y_0 < y < Y. \qquad (2.26)$$

An examination of the proof of Theorem 1 yields the

COROLLARY 1. *Suppose that h satisfies (2.4) for some constant $\alpha \geqslant 0$. Then any solution of Problem A which belongs to class $\mathscr{M}$ will depend continuously in L_2 on the prescribed Cauchy data.*

Let us remark that if we impose additional smoothness requirements on φ then a result similar to Theorem 1 holds even if $\alpha = 0$. The condition $\alpha > 0$ arose in treating the term

$$2 \iint_{D_t} (t - \eta)\, v_{,\eta} L\varphi \, dx \, d\eta \geqslant - \alpha \iint_{D_t} (t - \eta)\, v^2{}_{,\eta} \, dx \, d\eta$$

$$- \alpha^{-1} \iint_{D_t} (t - \eta)\, (L\varphi)^2 \, dx \, d\eta. \qquad (2.27)$$

However, integrating the left-hand side of (2.27) by parts before applying the arithmetic-geometric mean inequality one obtains instead

$$\iint_{D_t} (t - \eta)\, v_{,\eta} L\varphi \, dx \, d\eta = \int_{\substack{0 \\ y=-y_0}}^{1} vL\varphi \, dx + \iint_{D_t} vL\varphi \, dx \, d\eta$$

$$- \iint (t - \eta)\, v \frac{\partial}{\partial \eta}(L\varphi) \, dx \, d\eta$$

$$\geqslant - b_1 \int_{\substack{0 \\ y=-y_0}}^{1} v^2 \, dx - b_2 \int_{\substack{0 \\ y=-y_0}}^{1} (L\varphi)^2 \, dx$$

$$- b_3 \iint_{D_t} v^2 \, dx \, d\eta - b_4 \iint_{D_t} \left[\frac{\partial}{\partial \eta}(L\varphi)\right]^2 dx \, d\eta$$

$$- b_5 \iint_{D_t} (L\varphi)^2 \, dx \, d\eta. \qquad (2.28)$$

By adding the data terms

$$\int_{\substack{0 \\ y=-y_0}}^{1} (L\varphi)^2 \, dx \qquad \text{and} \qquad \iint_D \left[\frac{\partial}{\partial y}(L\varphi)\right]^2 dx \, dy$$

to the expression for Q given in (2.8) and proceeding as above one obtains the desired result for non-negative α.

For the sake of simplicity we have considered the case of a rectangular domain. The results obtained hold also for more general domains; in particular, they hold for domains for which the outward normal (n_x, n_y) satisfies the less restrictive condition that $n_y(n_y{}^2 + hn_x{}^2)$ is non-negative on the "sides."

Finally, we remark that in some cases it is sufficient to prescribe an L_2 bound for u on only part of D. For example, in the case of transonic flow with $h(0) = 0$, if $\alpha = 0$ then one needs only to prescribe an L_2 bound for u when

$0 \leqslant y \leqslant Y$ since the required bound for u when $-y_0 \leqslant y \leqslant 0$ can be determined from the data of the problem.

3. Let D be the rectangular domain defined in Section 2. In this section we show by means of an example that if, instead of Problem A, Cauchy data is prescribed on the line $y = Y$ and (2.4) is satisfied then even the imposition of the condition of uniform boundedness in $\bar{D}$ on the solution may not be sufficient to guarantee continuous dependence on the Cauchy data. This example shows also that (2.4) is a "best possible" condition for Problem A.

For the sake of simplicity we take $y_0 = 0$ and $Y = 1$. Then D is the domain $0 < x < 1$ and $0 < y < 1$. We consider the Tricomi equation in D, namely,

$$u_{,yy} + yu_{,xx} = 0. \tag{3.1}$$

Let us note that, setting $\alpha = 0$, (2.4) is satisfied for $0 \leqslant y \leqslant 1$. A particular solution of (3.1) is given by

$$u(x, y) = \varphi(y) \sin nx, \tag{3.2}$$

where φ is a solution of the ordinary differential equation

$$\frac{d^2\varphi}{dy^2} - n^2 y\varphi = 0. \tag{3.3}$$

Setting $\xi = n^{2/3}y$, Eq. (3.3) becomes

$$\varphi'' - \xi\varphi = 0, \tag{3.4}$$

where the prime denotes differentiation with respect to ξ. A particular solution of (3.4) is given by (see, e.g., [10])

$$\lambda(\xi) = \int_0^\infty \left[\cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\,\xi t\right)\right] \exp\left(-\frac{1}{2}\,\xi t - \frac{1}{3}\,t^3\right) dt. \tag{3.5}$$

If we set $\tau = \xi t$ then

$$\lambda(\xi) = \frac{1}{\xi}\int_0^\infty \left[\cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\,\tau\right)\right] \exp\left(-\frac{1}{2}\,\tau - \frac{1}{3\xi^3}\,\tau^3\right) d\tau, \tag{3.6}$$

from which it follows that $\lambda(\xi) = 0(\xi^{-1})$ and $\lambda'(\xi) = 0(\xi^{-2})$ as $\xi \to +\infty$. In fact, a more careful analysis [10] shows that $\lambda(\xi) \to 0$ as $\xi \to +\infty$ of order greater than any preassigned positive integer; that is, for any positive integer m

$$|\lambda(\xi)| \leqslant \frac{c}{\xi^m} \quad (\xi > 0), \tag{3.7}$$

where c is a positive constant. Let us define

$$u(x, y) = \lambda(n^{2/3}y) \sin nx. \tag{3.8}$$

We note that u is uniformly bounded in $\bar{D}$ with respect to n. By direct calculation one sees that the function u satisfies (3.1) in D and the boundary conditions

$$u(0, y) = u(1, y) = 0 \qquad 0 \leqslant y \leqslant 1, \tag{3.9}$$

$$u(x, 1) = \lambda(n^{2/3}) \sin nx \tag{3.10}$$

and

$$u_{,y}(x, 1) = n^{2/3}\lambda'(n^{2/3}) \sin nx, \tag{3.11}$$

as well as,

$$u_{,x}(x, 1) = n\lambda(n^{2/3}) \cos nx. \tag{3.12}$$

Therefore, since

$$\lambda(n^{2/3}) = O(n^{-(2m/3)}) \qquad \text{and} \qquad \lambda'(n^{2/3}) = O(n^{-4/3})$$

as $n \to +\infty$, it follows for sufficiently large n that u, $u_{,y}$ and $u_{,x}$ can be made arbitrarily small on $y = 1$. But

$$\begin{aligned} u(x, 0) &= \lambda(0) \sin nx \\ &= \tfrac{1}{2}\, 3^{-1/6}\, \Gamma(\tfrac{1}{3}) \sin nx \end{aligned} \tag{3.13}$$

and, hence, u is not arbitrarily small on $y = 0$. Therefore, if Cauchy data is prescribed on $y = 1$ and Dirichlet data is prescribed on $x = 0$ and $x = 1$, then the requirement that the solution of (3.1) be uniformly bounded in $\bar{D}$ is not sufficient to guarantee continuous dependence on the Cauchy data.

If one replaces y by $-y$ in the above discussion, then D is the domain $0 < x < 1$ and $-1 < y < 0$, and (3.1) becomes

$$u_{,yy} - yu_{,xx} = 0. \tag{3.14}$$

We now consider Problem A for Eq. (3.14) with $y_0 = 1$ and $Y = 0$. Let us note that the requirement (2.4) is not satisfied. Therefore, since the above discussion shows that uniform boundedness in $\bar{D}$ of the solution is not sufficient to imply continuous dependence on the Cauchy data, (2.4) may be considered as a "best possible" condition for Problem A.

4. In this section we consider the equation conjugate to (2.1), namely,

$$Hv \equiv v_{,xx} + (h^{-1}(y)\, v_{,y})_{,y} = 0. \tag{4.1}$$

Let R be the rectangular domain $-y_0 < y < Y$ and $0 < x < X$ where $y_0 \geqslant 0$ and $Y \geqslant 0$. We consider the following initial-boundary value problem:

PROBLEM B.

$$Hv = 0 \qquad \text{in} \qquad R \tag{4.2}$$

$$v \text{ prescribed on} \qquad y = -y_0 \qquad \text{and} \qquad y = Y \tag{4.3}$$

$$v, \frac{\partial v}{\partial x} \qquad \text{prescribed on} \qquad x = 0. \tag{4.4}$$

The function h is an arbitrary sufficiently smooth function of y only; it need not satisfy condition (2.4). The function v is said to belong to $\mathcal{N}$ if $h^{-1}v_{,y}$ (in the limiting sense) $\in C^1$ and if

$$\int_{-y_0}^{Y} v^2(x, y)\, dy \leqslant N \qquad 0 \leqslant x \leqslant X \tag{4.5}$$

is satisfied for some prescribed constant N.

We remark that if u is a solution of $Lu = 0$ then $u_{,y}$ is a solution of $Hu = 0$ and, hence, Problem B could be used to treat a mixed problem for u where Cauchy data is given on $x = 0$ and Neumann data is given on $y = -y_0$ and $y = Y$; in this case the condition that u belongs to $\mathcal{N}$ becomes

$$\int_{-y_0}^{Y} u^2{}_{,y}(x, y)\, dy \leqslant N \qquad 0 \leqslant x \leqslant X. \tag{4.6}$$

For the sake of convenience we consider for Problem B only the questions of uniqueness and continuous dependence on data. The basic *a priori* inequalities are again obtained by means of a convexity argument. Let ℓ_z denote for each z the line segment $x = z$ and $-y_0 \leqslant y \leqslant Y$. We define the functional $\mathscr{F}$ as

$$\mathscr{F}(v, x) = \log F(v, x) + x^2 \qquad 0 < x < X, \tag{4.7}$$

where

$$\begin{aligned} F(v, x) &= \int_{\ell_x} v^2\, dy + \int_{\ell_0} \mid h^{-1}v^2{}_{,y} - v^2{}_{,x} \mid dy \\ &\equiv \int_{\ell_x} v^2\, dy + Q^2. \end{aligned} \tag{4.8}$$

Let us show that $\mathscr{F}$ is a convex function of x. We note that

$$F' = 2 \int_{\ell_x} vv_{,x}\, dy \tag{4.9}$$

$$F'' = 2 \int_{\ell_x} v^2{}_{,x}\, dy + 2 \int_{\ell_x} vv_{,xx}\, dy. \tag{4.10}$$

If we assume that $v = 0$ on $y = -y_0$ and $y = Y$, then

$$\int_{\ell_x} vv_{,xx}\, dy = \int_{\ell_x} h^{-1}v^2{}_{,y}\, dy \tag{4.11}$$

and, hence,

$$F'' = 4 \int_{\ell_x} v^2{}_{,x}\, dy + 2 \int_{\ell_x} (h^{-1}v^2{}_{,y} - v^2{}_{,x})\, dy. \tag{4.12}$$

The last term in (4.12) can be expressed in terms of the Cauchy data by means of the identity

$$0 = \int_0^x \int_{\ell_\xi} v_{,\xi}[v_{,\xi\xi} + (h^{-1}v_{,y})_{,y}]\, d\xi\, dy$$

$$= \int_{\ell_x} (v^2_{,x} - h^{-1}v^2_{,y})\, dy - \int_{\ell_0} (v^2_{,x} - h^{-1}v^2_{,y})\, dy. \qquad (4.13)$$

It follows that

$$F^2\mathscr{F}'' \geqslant 2F'^2 - 2F\underset{\sim}{Q}^2 \geqslant 0. \qquad (4.14)$$

An application of Jensen's inequality to $\mathscr{F}$ yields

THEOREM 2. *Let v be a solution of Problem B which belongs to $\mathscr{N}$. Let N and Q be defined as in (4.5) and (4.8), respectively. Then there exists a computable constant K such that*

$$\int_{\ell_x} v^2\, dy \leqslant KN^{2x/X}\underset{\sim}{Q}^{2(1-x/X)}, \qquad 0 < x < X. \qquad (4.15)$$

In this section a rectangular domain was considered merely for simplicity of presentation. The results obtained hold also for more general domains.

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